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Finite Element Methods Are Not Always Optimal*

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Consider a regularly elliptic $2m$th order boundary value problem $Lu = f$ with $f \in H^r(\Omega)$, where $r \geq -m$. In previous work, we showed that the finite element method (FEM) using piecewise polynomials of degree k is asymptotically optimal when $k \geq 2m - 1 + r$. In this paper, we show that the FEM is *not* asymptotically optimal when this inequality is violated. However, there exists an algorithm, called the spline algorithm, which uses the same information as the FEM and *is* optimal. Moreover, the error of the finite element method can be arbitrarily larger than the error of the spline algorithm. We also obtain a necessary and sufficient condition for a Galerkin method (or a generalized Galerkin method) to be a spline algorithm.

1. INTRODUCTION

This paper deals with the optimal solution of $2m$th order regularly elliptic boundary value problems $Lu = f$ with $f \in H^r(\Omega)$ for a region $\Omega \in \mathbb{R}^N$ (see Section 2). We consider the variational form of such problems having homogeneous boundary conditions. We wish to solve such problems using information of cardinality at most n. (In this introduction, we have to use words such as information, cardinality, and algorithm without definition; they are defined rigorously in Section 3.)

In [11], we showed that the minimal energy-norm error of an algorithm using information of cardinality n is $\Theta(n^{-(m+r)/N})$ as $n \to \infty$.[1] Moreover, this minimal error is achieved by an n-evaluation finite element method (FEM) using piecewise polynomials of degree k, where $k \geq 2m - 1 + r$.

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[1] Here, and in what follows, we use the Ω- and Θ-notations of [5], as well as the usual O-notation. That is, $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

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In this paper,we determine what happens when the inequality $k \geq 2m - 1 + r$ is violated. (For example, we may be using a piecewise-linear finite element code for a second-order problem $Lu = f$ with $f \in H^1(\Omega)$.) In Section 4, we show that the error of the FEM is $\Theta(n^{-\mu/N})$ as $n \to \infty$, where $\mu = \min\{k + 1 - m, m + r\}$. Hence, $k \geq 2m - 1 + r$ is a necessary and sufficient condition for the FEM to be asymptotically optimal.

Suppose that $k < 2m - 1 + r$, so that the error of the FEM is no longer optimal. Is this nonoptimality because the finite element information (FEI) used by the FEM is inherently bad or is it because the FEM does not use its information well? We show that the latter is the case. More precisely, we consider the spline algorithm using FEI (see, e.g., [6; 10, Chap. 4]. It is known that the spline algorithm using given information has minimal error among all algorithms using that information. We show that the error of the spline algorithm using FEI is $\Theta(n^{-(r+m)/N})$ as $n \to \infty$, whether or not $k \geq 2m - 1 + r$. Hence, the spline algorithm using FEI is always an asymptotically optimal-error algorithm.

Since the spline algorithm has such strong optimal-error properties, we investigate it in more detail in Section 5. We show that the spline algorithm may be viewed as a generalized Galerkin method (i.e., one with different test and trial spaces). We give a condition that is necessary and sufficient for a given generalized Galerkin method to be a spline algorithm. This yields a condition that is necessary and sufficient for the FEM to be a spline algorithm. After showing an example of a FEM that is not a spline algorithm, we conjecture that no convergent FEM can be a spline algorithm.

Up to this point, we have mainly been concerned with minimal-error algorithms. In Section 6, we seek optimal-complexity algorithms for obtaining ϵ-approximations. That is, we seek algorithms whose error is at most ϵ, and whose cost is minimal among all algorithms having error at most ϵ. We find that the ϵ-complexity of our problem (i.e., the minimal cost of finding an ϵ-approximation) is $\Theta(\epsilon^{-N/(r+m)})$ as $\epsilon \to 0$. The cost of using the FEM to find an ϵ-approximation is $\Theta(\epsilon^{-N/\mu})$, where (as before) $\mu = \min\{k + 1 - m, m + r\}$. Hence the FEM is an almost-optimal-complexity algorithm iff $k \geq 2m - 1 + r$. Moreover, the penalty for using the FEM when $k < 2m - 1 + r$, rather than an optimal-complexity algorithm, is unbounded as $\epsilon \to 0$. On the other hand, the cost of using the spline algorithm to find an ϵ-approximation is $\Theta(\epsilon^{-N/(m+r)})$. Hence, the spline algorithm using FEI is always an almost-optimal-complexity algorithm. This means that if $k < 2m - 1 + r$ and if ϵ is sufficiently small, it is cheaper to use the spline algorithm for ϵ-approximation than the FEM. Although this is an asymptotic result, we show (via an example) that if $k < 2m - 1 + r$, then the spline algorithm can yield an ϵ-approximation more cheaply than the FEM, for moderate values of ϵ.

2. The Variational Boundary-Value Problem

In what follows, we use the standard notations and definitions (see, e.g., [2]) of multi-indices and of Sobolev spaces, inner products, and norms. Fractional- and negative-order Sobolev spaces are (respectively) defined by Hilbert-space interpolation [1, Chap. 2] and duality [7, Chap. 4].

Let $\Omega \subset \mathbb{R}^N$ be a bounded, simply connected, C^∞ region. Define the properly elliptic operator

$$Lv = \sum_{|\alpha|, |\beta| \le m} (-1)^{|\alpha|} D^\alpha \left(a_{\alpha\beta} D^\beta v \right) \tag{2.1}$$

(with real coefficients $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$ such that $a_{\alpha\beta} = a_{\beta\alpha}$) and a normal family of boundary operators

$$B_j v = \sum_{|\alpha| \le q_j} b_{j\alpha} D^\alpha v \qquad (0 \le j \le m-1) \tag{2.2}$$

(with real coefficients $b_{j\alpha} \in C^\infty(\partial\Omega)$), where

$$0 \le q_0 \le \cdots \le q_{m-1} \le 2m - 1, \tag{2.3}$$

which covers L on $\partial\Omega$. Setting

$$m^* = \min\{ j : q_j \ge m \}, \tag{2.4}$$

we additionally assume that

$$\{ q_j \}_{j=0}^{m^*-1} \cup \{ 2m - 1 - q_j \}_{j=m^*}^{m-1} = \{ 0, \ldots, m-1 \}. \tag{2.5}$$

(See [1, Chap. 3; 7, Chap. 5] for further definitions and illustrative examples.)

We are interested in approximating the variational solution of the boundary-value problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ B_j u &= 0 \quad \text{on } \partial\Omega \qquad (0 \le j \le m-1), \end{aligned} \tag{2.6}$$

where $f \in H^r(\Omega)$. To do this, we let

$$H_E^m(\Omega) = \left\{ v \in H^m(\Omega) \colon B_j v = 0 \; (0 \le j \le m^* - 1) \right\} \tag{2.7}$$

denote the space of $H^m(\Omega)$-functions satisfying the essential boundary conditions in (2.6). We define a symmetric, continuous bilinear form B on

$H_E^m(\Omega)$ by

$$B(v, w) = \sum_{|\alpha|, |\beta| \le m} \int_\Omega a_{\alpha\beta} D^\alpha v D^\beta w \qquad \forall v, w \in H_E^m(\Omega). \tag{2.8}$$

We additional assume that B is $H_E^m(\Omega)$-coercive, so that B is an inner product on $H_E^m(\Omega)$, yielding a norm $\|\cdot\|_B$ defined by

$$\|v\|_B = \sqrt{B(v, v)} \qquad \forall v \in H_E^m(\Omega), \tag{2.9}$$

which is equivalent to the usual norm $\|\cdot\|_m$ on $H_E^m(\Omega)$.

We may now define the *variational boundary problem* as follows. Let $r \ge -m$. For $f \in H^r(\Omega)$, we wish to find $u = Sf \in H_E^m(\Omega)$ such that

$$B(u, v) = (f, v)_0 = \int_\Omega fv \qquad \forall v \in H_E^m(\Omega). \tag{2.10}$$

From the Lax–Milgram theorem, S is a Hilbert space isomorphism of $H^{-m}(\Omega)$ onto $H_E^m(\Omega)$, and so S: $H^r(\Omega) \to H_E^m(\Omega)$ is a bounded linear injection.

3. Information and Algorithms

In this section, we define some of the concepts (which are taken from [10]) that were mentioned in the Introduction. They are illustrated for the finite element method (FEM) and the finite element information (FEI) it uses. For further explanation of the concepts of this section, the reader should consult [10].

We wish to approximate Sf for $f \in F$, with F denoting the unit ball of the Sobolev space $H^r(\Omega)$, where $r \ge -m$. (Our restriction to the unit ball of $H^r(\Omega)$ may be considered a normalization; see [10] for further discussion.) Hence, we are trying to approximate the transformation S: $F \to H_E^m(\Omega)$. To do this, we must know something about each problem element $f \in F$. In this paper, we assume that we know *information* N of the form

$$Nf = \begin{bmatrix} \lambda_1(f) \\ \vdots \\ \lambda_n(f) \end{bmatrix} \qquad \forall f \in F. \tag{3.1}$$

Here, $\lambda_1, \ldots, \lambda_n$ are linearly independent linear functionals on $H^r(\Omega)$, and we say that n is the *cardinality* of N.

An *algorithm* φ *using* N is then a mapping $\varphi\colon \mathbb{R}^n \to H_E^m(\Omega)$. The error of such an algorithm is then defined to be

$$e(\varphi, N) = \sup_{f \in F} \|Sf - \varphi(Nf)\|_B. \tag{3.2}$$

One of our main goals will be to determine the minimal error among all algorithms using given information N. That is, we wish to find

$$r(N) = \inf_{\varphi} e(\varphi, N). \tag{3.3}$$

For geometrical reasons, we call $r(N)$ the *radius of information*. An algorithm φ_N such that

$$e(\varphi_N, N) = r(N) \tag{3.4}$$

is said to be an *optimal-error algorithm* using N.

EXAMPLE 3.1. We illustrate these ideas by considering the finite element method. Let $\{\mathscr{S}_n\}_{n=1}^{\infty}$ be a quasi-uniform family of finite element subspaces of degree k. That is, $\mathscr{S}_n$ is an n-dimensional subspace of $H_E^m(\Omega)$ consisting of piecewise polynomials of degree k over a triangulation T_n of Ω, where $\{\mathscr{T}_n\}_{n=1}^{\infty}$ is quasi-uniform [7, pg. 132]. (Of course, since Ω is C^∞, it will not generally be the case that $\mathscr{S}_n \subseteq H_E^m(\Omega)$. In this paper, we will ignore this source of error. If necessary, this error may be removed by using isoparametric elements as in [3].)

We now define the *finite element method* (FEM) using $\{\mathscr{S}_n\}_{n=1}^{\infty}$. Let $\{s_1, \ldots, s_n\}$ be a basis for $\mathscr{S}_n$. For $f \in F$, we evaluate the inner products

$$N_n^* f = \begin{bmatrix} (f, s_1)_0 \\ \vdots \\ (f, s_n)_0 \end{bmatrix} \tag{3.5}$$

We then choose $u_n \in \mathscr{S}_n$ such that

$$B(u_n, s_i) = (f, s_i)_0 \qquad (1 \le i \le n). \tag{3.6}$$

The FEM yields u_n^*, which depends only on $N_n^* f$; we write $u_n^* = \varphi_n^*(N_n^* f)$. We call N_n^* *finite element information* (FEI) of *cardinality* n.

We discuss some properties of the FEM. It is well known that for any n, there exists a unique solution $u_n^* \in \mathscr{S}_n$ to (3.6), and that

$$\|Sf - \varphi_n^*(N_n^* f)\|_B = \|u - u_n^*\|_B = \inf_{s \in \mathscr{S}_n} \|u - s\|_B. \tag{3.7}$$

Moreover, there exists a positive constant C (independent of f, u, n, and

u_n) such than

$$\|Sf - \varphi_n^*(N_n^* f)\|_B = \|u - u_n^*\|_B \le Cn^{-\mu/N}\|f\|_r, \tag{3.8}$$

where

$$\mu = \min\{k + 1 - m, m + r\}. \tag{3.9}$$

Hence, the error of the finite element method satisfies the inequality

$$e(\varphi_n^*, N_n^*) \le Cn^{-\mu/N}. \tag{3.10}$$

Moreover, the results in [9] suggest that this inequality should be two-sided. We will show that this is indeed the case in the next section. We will compute the radius of finite element information, and show that

$$r(N_n^*) = \Theta(n^{-(m+r)/N}) \qquad \text{as } n \to \infty. \tag{3.11}$$

Hence, it will follow that the FEM is an almost-optimal-error algorithm using N_n^* iff $k \ge 2m - 1 + r$. □

Just as we can ask which algorithm makes optimal use of given information, one can also ask which information of given cardinality is best. Let

$$r(n) = \inf\{r(N)\colon N \text{ is information of cardinality at most } n\} \tag{3.12}$$

denote the nth *minimal radius of information*. Information N_n such that

$$r(N_n) = r(n) \tag{3.13}$$

is said to be nth *optimal information*. An optimal-error algorithm using nth optimal information is said to be an nth *minimal-error algorithm*.

EXAMPLE 3.1 (*continued*). From [11], we have

$$r(n) = \Theta(n^{-(m+r)/N}) \qquad \text{as } n \to \infty. \tag{3.14}$$

Using the results in the next section, it will then follow that N_n^* is almost-optimal information. Moreover, we will see that the FEM is an almost-minimal error algorithm iff $k \ge 2m - 1 + r$.

4. ERROR OF THE FEM AND RADIUS OF FEI

In this section, we derive a lower bound for the FEM of degree k. Using this lower bound, we show that the FEM is an almost-minimal error algorithm iff $k \ge 2m - 1 + r$. We also show that the spline algorithm using FEI is always an almost-minimal error algorithm, regardless of the value of k. Hence, the reason the FEM (of too small a degree) does not have almost-minimal error is that it uses its information poorly; there exists an algorithm using the same information that has almost-optimal error.

Recall that $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is a sequence of n-dimensional subspaces of $H_E^m(\Omega)$, and that $\mathcal{S}_n$ consists of piecewise polynomials of degree k over the triangulation $\mathcal{T}_n$ of Ω. We first show

LEMMA 4.1. $k \geq m$.

Proof. Suppose on the contrary that $k \leq m - 1$. Let $P_k(\Omega)$ denote the space of polynomials of degree at most k over Ω. Choose $n > \dim P_k(\Omega)$. We claim that $\mathcal{S}_n \subseteq P_k(\Omega)$. To see this, let $s \in \mathcal{S}_n$, and let K_1 and K_2 be adjacent elements in the triangulation $\mathcal{T}_n$. Let s_1 and s_2 respectively denote the restriction of s to K_1 and K_2. Since an obvious extension of [2, Theorem 4.2.1] yields that $s \in C^{m-1}(K_1 \cup K_2)$ and since s_1 and s_2 are polynomials of degree at most $k \leq m - 1$, it is easy to check that $s_1 = s_2$ on $K_1 \cup K_2$. From this, it follows that s is a polynomial, and not merely a piecewise polynomial, on $K_1 \cup K_2$. Repeating this argument to include all elements of the triangulation, we now find that $s \in P_k(\Omega)$. Thus, $\mathcal{S}_n \subseteq P_k(\Omega)$. Since $\dim \mathcal{S}_n = n > \dim P_k(\Omega)$, this is impossible. □

We are now ready to establish the sharpness of the usual error estimate for the FEM, generalizing the work of [9]. Recall that r is the smoothness of the class of problem elements, and that k is the degree of the FEM. Also, recall that we have defined

$$\mu = \min\{k + 1 - m, m + r\} \tag{4.1}$$

in Section 3.

THEOREM 4.1. *The error of the FEM of degree k is*

$$e(\varphi_n^*, N_n^*) = \Theta(n^{-\mu/N}) \qquad \text{as } n \to \infty.$$

Hence, the FEM is an almost-minimal error algorithm iff

$$k \geq 2m - 1 + r.$$

Proof. We need only establish the first part of this theorem, since the second part follows immediately from this estimate and from (3.14).

From (3.10), we have the upper bound

$$e(\varphi_n^*, N_n^*) = O(n^{-\mu/N}) \qquad \text{as } n \to \infty. \tag{4.2}$$

It remains to prove the lower bound

$$e(\varphi_n^*, N_n^*) = \Omega(n^{-\mu/N}) \qquad \text{as } n \to \infty. \tag{4.3}$$

Since N_n^* is information of cardinality n, we may use (3.14) to find that

$$e(\varphi_n^*, N_n^*) \geq r(n) = \Omega(n^{-(m+r)/N}) \qquad \text{as } n \to \infty. \tag{4.4}$$

Hence, it remains to show that

$$e(\varphi_n^*, N_n^*) = \Omega(n^{-(k+1-m)/N}) \qquad \text{as } n \to \infty, \tag{4.5}$$

since the lower bound (4.3) follows immediately from (4.4) and (4.5).

To prove (4.5), let Ω^0 be the interior of a hypercube whose closure is contained in Ω. For each n, let $\mathscr{T}_n^0$ denote the elements in the triangulation $\mathscr{T}_n$ which are contained in the closure of Ω^0. Finally, we let Ω_n denote the interior of the union of the elements in $\mathscr{T}_n^0$.

We now choose any function $u \in H_E^m(\Omega)$ such that

$$u(x) = \frac{1}{(k+1)!} x_1^{k+1} \qquad \forall x \in \overline{\Omega^0}. \tag{4.6}$$

Let $K \in \mathscr{T}_n^0$. We claim that there is a constant $C > 0$, independent of K and n, such that

$$\inf_{s \in P_k(K)} |u - s|_{m,K}^2 \geq C(\operatorname{vol} K)^{2(k+1-m)/N+1} \tag{4.7}$$

(Here, and in what follows, we use the notation of [2, Chap. 3].) To prove (4.7), let $\hat{K}$ denote a "reference element" which is independent of n and K, so that K is the affine image of $\hat{K}$. Using the scaling techniques of [2, Theorem 3.1.2], the proof of (4.7) may be reduced to showing that there is a constant $\hat{C}$, depending only on k, m, and $\hat{K}$, such that

$$\inf_{\hat{s} \in P_k(\hat{K})} |\hat{v} - \hat{s}|_{m,\hat{K}} \geq \hat{C} |\hat{v}|_{k+1,\hat{K}} \qquad \forall \hat{v} \in P_{k+1}(\hat{K}). \tag{4.8}$$

But the left-hand and right-hand sides of (4.8) are seminorms of $\hat{v} \in P_{k+1}(\hat{K})$. Since $k \geq m$ by Lemma 4.1, these two seminorms have the same kernel, namely, the space $P_k(\hat{K})$. Inequality (4.8) now follows immediately from the finite-dimensionality of $P_{k+1}(\hat{K})$; this (in turn) establishes the inequality (4.7).

Since $\mathscr{T}_n^0 \subseteq \mathscr{T}_n$, we may use (4.7) to find that

$$\begin{aligned} \inf_{s \in \mathscr{S}_n} |u - s|_m^2 &\geq \sum_{K \in \mathscr{T}_n^0} \inf_{s \in P_k(K)} |u - s|_{m,K}^2 \\ &\geq C \sum_{K \in \mathscr{T}_n^0} (\operatorname{vol} K)^{2(k+1-m)/N+1} \end{aligned} \tag{4.9}$$

Since

$$\sum_{K \in \mathcal{T}_n^0} \operatorname{vol} K = \operatorname{vol} \overline{\Omega_n}, \tag{4.10}$$

we may use calculus to find that

$$\sum_{K \in \mathcal{T}_n^0} (\operatorname{vol} K)^{2(k+1-m)/N+1} \geq \left(\frac{\operatorname{vol} \overline{\Omega_n}}{\# \mathcal{T}_n^0} \right)^{2(k+1-m)/N} \tag{4.11}$$

From the quasi-uniformity of $\{\mathcal{T}_n\}_{n=1}^{\infty}$, there is an index n_0 such that

$$\operatorname{vol} \overline{\Omega_n} \geq \tfrac{1}{2} \operatorname{vol} \overline{\Omega^0} \qquad \forall n \geq n_0. \tag{4.12}$$

From (4.9)–(4.12), we find that there is a constant $C > 0$, which is independent of n, such that

$$\inf_{s \in \mathcal{S}_n} |u - s|_m \geq C\left(\# \mathcal{T}_n^0 \right)^{-(k+1-m)/N} \qquad \forall n \geq n_0. \tag{4.13}$$

We next claim that there is a constant $C > 0$, independent of n, such that

$$\# \mathcal{T}_n^0 \leq Cn. \tag{4.14}$$

To prove (4.14) in the case $m = 0$, we merely note that the set of characteristic functions of $\mathcal{T}_n^0$-elements has size $\# \mathcal{T}_n^0$ and is a linearly independent subset of the n-dimensional space $\mathcal{S}_n$. We now consider the case $m \geq 1$. Let $\mathcal{S}_n(\overline{\Omega_n})$ denote the restrictions of $\mathcal{S}_n$-functions to $\overline{\Omega_n}$. In the case $N = 1$, we may count free parameters and use the inequality $\dim \mathcal{S}_n(\overline{\Omega_n}) \leq \dim \mathcal{S}_n = n$ to see that

$$\# \mathcal{T}_n^0 \leq \frac{n - m}{k + 1 - m}. \tag{4.15}$$

For the case $N \geq 2$, we let $v(\mathcal{T}_n^0)$ denote the number of vertices in $\mathcal{T}_n^0$. Using the quasi-uniformity of $\{\mathcal{T}_n\}_{n=1}^{\infty}$, we see that there is a constant C, independent of n, such that

$$\# \mathcal{T}_n^0 \leq Cv\left(\mathcal{T}_n^0 \right). \tag{4.16}$$

We need only show that there is a constant C, independent of n, such that

$$v\left(\mathcal{T}_n^0 \right) \leq C \dim \mathcal{S}_n\left(\overline{\Omega_n} \right). \tag{4.17}$$

In the case $N = 2$, inequality (4.16) follows from [12, Theorem 1]; the case $N > 2$ may be reduced to the case $N = 2$ by considering restrictions of functions in $\mathcal{S}_n(\overline{\Omega_n})$ to 2-faces of simplices $K \in \mathcal{T}_n^0$.

Combining (4.13) and (4.14) and using the inequality $\|\cdot\|_m \geq |\cdot|_m$, we see that there is a positive constant C such that for any $n \geq n_0$, there exists $u \in H_E^m(\Omega)$ such that

$$\inf_{s \in \mathcal{S}_n} \|u - s\|_m \geq C n^{-(k+1-m)/N} \qquad \forall n \geq n_0. \tag{4.18}$$

Since u is a nonzero element of $H^{2m+r}(\Omega) \cap H_E^m(\Omega)$, we see that Lu is a nonzero element of $H^r(\Omega)$. Let

$$f^* = \frac{1}{\|Lu\|_r} Lu. \tag{4.19}$$

Then $f^* \in F$. Since $\mathcal{S}_n$ is a subspace of $H_E^m(\Omega)$ and the norms $\|\cdot\|_m$ and $\|\cdot\|_B$ are equivalent, we may combine (3.2), (3.7), and (4.19) to find that (4.5) holds, as claimed. □

Hence, the FEM of degree k is not an almost-minimal error algorithm unless $k \geq 2m - 1 + r$. There are two reasons why the error of the FEM is not almost-minimal. Either

(1) the FEM does not make good use of its information, and there is another algorithm using FEI whose error is almost-minimal, regardless of whether $k \geq 2m - 1 + r$, or

(2) the FEM is an almost-optimal error algorithm using FEI, and FEI is not strong enough information to admit an almost-minimal error algorithm.

It turns out that (1) is the reason the FEM (of too-low a degree) does not have almost-minimal error.

To see this, we will show that the spline algorithm using FEI has almost-minimal error. Recall that N_n^* is finite element information based on the n-dimensional space $\mathcal{S}_n$ of piecewise polynomials of degree k. Let P: $H^r(\Omega) \to H^r(\Omega)$ denote the orthogonal projector onto $(\ker N_n^*)^\perp$. Then the *spline algorithm* φ_n^s using N_n^* is defined by

$$\varphi_n^s(N_n^* f) = SPf \qquad \forall f \in F. \tag{4.20}$$

(See [6, 10] for further discussion.)

We then have

THEOREM 4.2. *The spline algorithm using FEI is an optimal-error algorithm using FEI and is an almost-minimal error algorithm. That is,*

$$e(\varphi_n^s, N_n^*) = r(N_n^*) = \Theta(r(n)) = \Theta(n^{-(r+m)/N}) \qquad \textit{as } n \to \infty.$$

Proof. The spline algorithm using information N is always an optimal error algorithm using N; see, e.g., [6; 10, Chap. 4] for details. Hence,

$$e(\varphi_n^s, N_n^*) = r(N_n^*). \tag{4.21}$$

From (3.14), we have the lower bound

$$r(N_n^*) \geq r(n) = \Theta(n^{-(r+m)/N}) \qquad \text{as } n \to \infty. \tag{4.22}$$

Hence, it remains to prove the upper bound

$$r(N_n^*) = O(n^{-(r+m)/N}) \qquad \text{as } n \to \infty. \tag{4.23}$$

To do this, let $z \in F \cap \ker N_n^*$. Since $k \geq m$ by Lemma 4.1, we may use [1, Theorem 4.1.1], along with the norm-equivalence of $\|\cdot\|_m$ and $\|\cdot\|_B$, to see that there exists $s_n \in \mathscr{S}_n$ such that

$$\|Sz - s\|_{-r} \leq Cn^{-(m+r)/N}\|Sz\|_B, \tag{4.24}$$

the constant C being independent of n and z. Since $s_n \in \mathscr{S}_n$, we have $(z, s)_0 = 0$. Since z is in the unit ball of $H^r(\Omega)$, we may use (2.8), (2.9), and (4.24) to find that

$$\begin{aligned} \|Sz\|_B^2 &= B(Sz, Sz) = (z, Sz)_0 = (z, Sz - s_n)_0 \leq \|Sz - s\|_{-r}\|z\|_r \\ &\leq Cn^{-(m+r)/N}\|Sz\|_B. \end{aligned} \tag{4.25}$$

Since z is an arbitrary element of $F \cap \ker N_n^*$, we may use [10, Theorem 4.5.1] and (4.25) to find that

$$r(N_n^*) = \sup_{f \in F \cap \ker N_n^*} \|Sz\|_B \leq Cn^{-(r+m)/N}, \tag{4.25}$$

completing the proof of the theorem. □

Hence the information N_n^* is always nth almost-optimal information. Moreover, the FEM of degree k is an almost-optimal error algorithm using FEI iff $k \geq 2m - 1 + r$.

Remark 4.1. In Section 2, we made some stringent assumptions about the smoothness of Ω and the coefficients of the differential operators $L, B_0, \dots, B_{m-1}$, in order to guarantee that the "shift theorem" (see, e.g., [1, Chap. 3; 7, Chap. 5]) holds for our problem. If these smoothness assumptions are violated, then the shift theorem no longer holds. It is easy to check that for such problems, the nth minimal radius remains unchanged. However, the FEM will no longer be an almost-minimal error algorithm, because the error estimate of the FEM depends critically on the shift theorem. Since the proof of the error bound of the spline algorithm did not depend on a shift theorem, the spline algorithm is still an almost-minimal error algorithm in this nonsmooth case. For example, consider Poisson's equation in two dimensions, with Dirichlet data on part of the boundary, and Neumann data on the remainder. Following [4], we find that the error of the FEM of arbitrary degree is $\Omega(n^{-3/(2N)})$ as $n \to \infty$, while the error of the spline algorithm is $\Theta(n^{-(r+1)/2})$, as is the nth minimal radius. Hence, the FEM has almost-minimal error for this problem with mixed boundary conditions iff $r < \frac{1}{2}$, whereas the spline algorithm using FEI always has almost-minimal error. □

5. Spline Algorithms and Generalized Galerkin Methods

In the previous section, we introduced the spline algorithm, which is always an optimal-error algorithm using given information. We saw that the spline algorithm using FEI is always an almost-minimal error algorithm, regardless of the degree of the finite element space. In this section, we show that the spline algorithm is a generalized Galerkin method using different spaces of test and trial functions, and establish a necessary and sufficient condition or a Galerkin method to be a spline algorithm. Based on a simple example, we conjecture that on convergent FEM is a spline algorithm.

We first describe generalized Galerkin methods. Let $\{s_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^n$ each be sets of linearly independent functions in $H_E^m(\Omega)$. Let

$$\mathscr{S} = \operatorname{lin}\{s_i\}_{i=1}^n \qquad \text{and} \qquad \mathscr{T} = \{t_i\}_{i=1}^n \tag{5.1}$$

respectively denote the subspaces of *test* and *trial* functions. Define *Galerkin information* $N_{\mathscr{S}}$ based on $\mathscr{S}$ by

$$N_{\mathscr{S}} f = \begin{bmatrix} (f, s_1)_0 \\ \vdots \\ (f, s_n)_0 \end{bmatrix} \qquad \forall f \in F. \tag{5.2}$$

Then the *generalized Galerkin method* $\varphi_{\mathscr{S},\mathscr{T}}$ based on $\mathscr{S}$ and $\mathscr{T}$ is given by

$$\varphi_{\mathscr{S},\mathscr{T}}(N_{\mathscr{S}} f) = u_{\mathscr{S},\mathscr{T}}, \tag{5.3}$$

where $u_{\mathscr{S},\mathscr{T}} \in \mathscr{T}$ satisfies

$$B(u_{\mathscr{S},\mathscr{T}}, s) = (f, s)_0 \qquad \forall f \in F. \tag{5.4}$$

When $\mathscr{T} = \mathscr{S}$, we write $\varphi_{\mathscr{S}}$ instead of $\varphi_{\mathscr{S},\mathscr{T}}$; the algorithm $\varphi_{\mathscr{S}}$ is the *Galerkin method* based on the subspace $\mathscr{S}$.

Remark 5.1. Of course, the FEM φ_n^* is a Galerkin method with $\mathcal{S} = \mathcal{S}_n$, where $\mathcal{S}_n$ is the n-dimensional subspace consisting of piecewise polynomials of degree k, as described in Section 3.

In what follows, we let $S^*\colon H_E^m(\Omega) \to H^r(\Omega)$ denote the adjoint to S, remembering that $H_E^m(\Omega)$ is a Hilbert space under the inner product given by the bilinear form B. Hence, (2.9) yields

$$(g, v)_0 = B(Sg, v) = (g, S^*v)_r \qquad \forall v \in H_E^m(\Omega),\ g \in H_r(\Omega). \quad (5.5)$$

We first give a representation formula for the spline algorithm using Galerkin information $N_{\mathcal{S}}$ and for the generalized Galerkin algorithm based on the spaces $\mathcal{S}$ and $\mathcal{T}$ of test and trial functions.

LEMMA 5.1. *Suppose that the respective bases $\{s_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^n$ of the subspaces $\mathcal{S}$ and $\mathcal{T}$ are chosen so that*

$$\left(S^*s_j, S^*s_i\right)_0 = \delta_{i,j} \qquad (1 \le i, j \le n) \quad (5.6)$$

and

$$B\left(t_j, s_i\right) = \delta_{i,j} \qquad (1 \le i, j \le n) \quad (5.7)$$

hold. Then the spline algorithm $\varphi_{\mathcal{S}}^s$ using $N_{\mathcal{S}}$ has the form

$$\varphi_{\mathcal{S}}^s(N_{\mathcal{S}}f) = \sum_{j=1}^n (f, s_j)_0 SS^*s_j,$$

and the generalized Galerkin algorithm $\varphi_{\mathcal{S},\mathcal{T}}$ has the form

$$\varphi_{\mathcal{S},\mathcal{T}}(N_{\mathcal{S}}f) = \sum_{j=1}^n (f, s_j)_0 t_j.$$

Proof. Using (2.9), it is straightforward to check that $S^*\mathcal{S} = (\ker N_{\mathcal{S}})^{\perp}$. Using the orthonormality of $\{S^*s_i\}_{i=1}^n$, we find that the orthogonal projector $P\colon H^r(\Omega) \to H^r(\Omega)$ is given by

$$Pf = \sum_{j=1}^n \left(f, s_j\right)_0 S^*s_j \qquad \forall f \in H^r(\Omega). \quad (5.8)$$

Since

$$\varphi_{\mathcal{S}}^s(N_{\mathcal{S}}f) = SPf \qquad \forall f \in F, \quad (5.9)$$

we now have the formula for the spline algorithm. The formula for the generalized Galerkin algorithm follows immediately from its definition and from the biorthogonality of the bases for $\mathcal{S}$ and $\mathcal{T}$. □

We now give the main result of this section, which tells us the unique choice of the trial function space $\mathcal{T}$ (corresponding to the given test function space $\mathcal{S}$) for which the generalized Galerkin method is the spline method.

THEOREM 5.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be n-dimensional subspaces of $H_E^m(\Omega)$. Then the following are equivalent*:

(1) *The generalized Galerkin algorithm is the spline algorithm, i.e.*, $\varphi_{\mathcal{S},\mathcal{T}} = \varphi_{\mathcal{S}}^s$.

(2) $\mathcal{T} = SS^*\mathcal{S}$.

Proof. Suppose first that (1) holds. Choose bases $\{s_i\}_{i=1}^n$ for $\mathcal{S}$ and $\{t_i\}_{i=1}^n$ for $\mathcal{T}$ such that (5.6) and (5.7) hold, so that the representation formulas of Lemma 5.1 hold. Using (5.5), we have

$$\begin{aligned} t_i &= \sum_{j=1}^n (S^*s_i, s_j)_0 t_j = \varphi_{\mathcal{S},\mathcal{T}}(N_{\mathcal{S}}S^*s_i) \\ &= \varphi_{\mathcal{S}}^s(N_{\mathcal{S}}S^*s_i) = \sum_{j=1}^n (S^*s_i, s_j)_0 SS^*s_j = SS^*s_i \end{aligned} \tag{5.10}$$

for $1 \le i \le n$, which implies (2).

Now suppose that (2) holds. Choose a basis $\{s_i\}_{i=1}^n$ for $\mathcal{S}$ such that (5.6) holds. Let

$$t_i = SS^*s_i \quad (1 \le i \le n). \tag{5.11}$$

Then (2) and injectivity of SS^* show that $\{t_i\}_{i=1}^n$ is a basis for $\mathcal{T}$. Using (5.5) and (5.6), we easily find that (5.7) holds. Using (5.6), (5.7), and (5.11), we see that the representation formulas of Lemma 5.1 imply that $\varphi_{\mathcal{S},\mathcal{T}} = \varphi_{\mathcal{S}}^s$, establishing (1). □

Hence, given any finite-dimensional subspace $\mathcal{S}$ of $H_E^m(\Omega)$, we see how to choose the unique subspace $\mathcal{T}$ of $H_E^m(\Omega)$ with $\dim \mathcal{T} = \dim \mathcal{S}$ such that $\varphi_{\mathcal{S}}^s = \varphi_{\mathcal{S},\mathcal{T}}$. However, the most natural choice of subspaces to pick $\mathcal{T} = \mathcal{S}$, so that we get the standard Galerkin method $\varphi_{\mathcal{S}}$. When is the standard Galerkin method $\varphi_{\mathcal{S}}$ the spline algorithm $\varphi_{\mathcal{S}}^s$?

THEOREM 5.2. *Let $\mathcal{S}$ be an n-dimensional subspace of $H_E^m(\Omega)$. Then the following are equivalent*:

(1) $\varphi_{\mathcal{S}} = \varphi_{\mathcal{S}}^s$.

(2) $\mathcal{S} = SS^*\mathcal{S}$.

(3) $\mathcal{S}$ *is an eigenspace of* SS^*.

(4) $\mathcal{S} = S\mathcal{F}$, *where* $\mathcal{F}$ *is an n-dimensional invariant subspace* (*or, equivalently, an n-dimensional eigenspace*) *of* S^*S.

Proof. The equivalence of the first two conditions is immediate from Theorem 5.1. Using a little simple linear algebra, the remainder of the proof follows immediately. □

We now illustrate Theorem 5.2 by two examples. In the first example, the Galerkin method is always a spline algorithm.

EXAMPLE 5.1. Let $r = -m$. The S is the Riesz map, which is an isometric isomorphism of $H^{-m}(\Omega)$ (under the norm $\|S\cdot\|_B$, which is equivalent to $\|\cdot\|_{-m}$) onto $H_E^m(\Omega)$; see [7, Sect. 4.4]. Hence $SS^* = I$, the identity map on $H_E^m(\Omega)$,and so $\mathcal{S} = SS^*\mathcal{S}$ for *any* subspace $\mathcal{S}$ of $H_E^m(\Omega)$. So when $r = -m$, the Galerkin method is always the spline algorithm, no matter what the choice of $\mathcal{S}$. Of course since $r = -m$, (3.14) shows that $\lim_{n\to\infty} r(n) \neq 0$. Hence the problem is not convergent, i.e., there is no convergent sequence of algorithms, each of which uses finite information (see also [10, Corollary 2.5.1]. □

In our second example, we exhibit a finite element method that is not a spline algorithm. This example is of particular interest because it gives an instance of an FEM that has optimal worst-case error (to within a constant) yet is not a spline algorithm.

EXAMPLE 5.2. We consider the L_2-approximation problem for H^1-functions on the unit interval $(0,1)$. Choose $N = 1$, $m = 0$, $r = 1$, and let S: $H^1(0,1) \to L_2(0,1)$ be the canonical injection

$$Su = u \qquad \forall u \in H^1(0,1). \tag{5.12}$$

The variational form of this problem is to find, for $f \in H^1(0,1)$, a function $u = Sf \in L_2(0,1)$ for which

$$(u, v)_0 = (f, v)_0 \qquad \forall v \in L_2(0,1). \tag{5.13}$$

(That is, the bilinear form B in (2.8) is merely the L_2-inner product.) Of course, $u = f$.

We let $\mathcal{S}_n$ be an n-dimensional subspace of $L_2(0,1)$ consisting of piecewise constants, so that $k = 0$. Let

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1 \tag{5.14}$$

be a partition of $(0,1)$. Then $\mathcal{S}_n$ is the span of the functions $s_1, \ldots, s_n$, where for $1 \le i \le n$,

$$s_i(x) = \delta_{ij} \qquad \forall x \in [x_{j-1}, x_j] \qquad (1 \le j \le n). \tag{5.15}$$

We next determine the space $SS^*\mathcal{S}_n$. Integrating by parts, we find that for any $s \in L_2(0,1)$, the function $w = SS^*s$ is the (weak) solution to the problem

$$\begin{aligned} -w'' + w &= s \quad \text{in } (0,1) \\ w'(0) = w'(1) &= 0, \end{aligned} \tag{5.16}$$

so that

$$w(x) = \frac{\int_0^1 s(\xi)\cosh(1-\xi)\, d\xi}{\sinh 1} \cosh x - \int_0^x s(\xi)\sinh(x-\xi)\, d\xi. \tag{5.17}$$

Hence, $SS^*\mathscr{S}_n$ is the span of the functions $w_1, \ldots, w_n$, where

$$\begin{aligned} w_i(x) = {} & \frac{\sinh(1-x_{i-1}) - \sinh(1-x_i)}{\sinh 1} \cosh x \\ & - \cosh(x - x_{i-1}) + \cosh(x - x_i) \qquad \text{for } 1 \le i \le n. \end{aligned} \tag{5.18}$$

Since none of the w_i is piecewise constant on $(0,1)$, we have $w_i \notin \mathscr{S}_n$. Hence $SS^*\mathscr{S}_n \neq \mathscr{S}_n$.

Using Theorem 5.2, we see that the FEM is not the spline algorithm. This is especially interesting, since this FEM is a quasi-minimal error algorithm. □

Examples 5.1 and 5.2 suggest the following

Conjecture 5.1. Let $r > -m$, and let $\mathscr{S}_n$ be a finite-element subspace of $H_E^m(\Omega)$. Then the FEM using $\mathscr{S}_n$ is not a spline algorithm. □

Clearly the results of Section 4 imply that Conjecture 5.1 holds when $k < 2m - 1 + r$. Hence, it remains to prove the conjecture for the case $k \ge 2m - 1 + r$.

6. Complexity Analysis

In this section, we discuss the complexity (minimal cost) of finding ϵ-approximations to the solution of the variational boundary-value problem, as well as the penalty for using the FEM when $k < 2m - 1 + r$.

The *cost* of an algorithm φ using information N is defined via the model of computation discussed in [10, Chap. 5]. That is, we assume that any linear functional required by φ can be evaluated with finite cost c, and that the cost of an arithmetic operation is unity. We denote the cost of an algorithm φ using information N by $\text{cost}(\varphi, N)$.

Let $\epsilon > 0$. An algorithm φ using information N produces an *ϵ-approximation* to the problem if

$$e(\varphi, N) \le \epsilon. \tag{6.1}$$

We then define, for $\epsilon > 0$, the *ϵ-complexity* $\mathrm{COMP}(\epsilon)$ of the problem to be

$$\mathrm{COMP}(\epsilon) = \inf\{\mathrm{cost}(\varphi, N) \colon \varphi \text{ is an algorithm using } N \text{ and } e(\varphi, N) < \epsilon\}. \tag{6.2}$$

For $\epsilon > 0$, an algorithm φ_ϵ using information N_ϵ for which

$$e(\varphi_\epsilon) \le \epsilon \qquad \text{and} \qquad \mathrm{cost}(\varphi_\epsilon, N_\epsilon) = \mathrm{COMP}(\epsilon) \tag{6.3}$$

is said to be an *optimal complexity algorithm* for finding an ϵ-approximation.

Remark 6.1. Note that we distinguish between the cost of an algorithm and the complexity of the problem. An optimal complexity algorithm for finding an ϵ-approximation is an algorithm which produces an ϵ-approximation with minimal cost. □

Recall that φ_n denotes the FEM of degree k using the FEI N_n based on the finite element subspace $\mathcal{S}_n$. Let

$$\mathrm{FEM}(\epsilon) = \inf\{\mathrm{cost}(\varphi_n, N_n) : n \text{ is an index such that } e(\varphi_n, N_n) \le \epsilon\} \tag{6.4a}$$

denote the cost of solving the problem using the FEM. Also recalling that φ_n^s denotes the spline algorithm using the FEI N_n, we let

$$\mathrm{SPLINE}(\epsilon) = \inf\{\mathrm{cost}(\varphi_n^s, N_n) : n \text{ is an index such that } e(\varphi_n^s, N_n) \le \epsilon\} \tag{6.4b}$$

denote the cost of solving the problem using the spline algorithm.

Using the results of Section 4 and of [10, Chap. 5], we have

THEOREM 6.1. (1) *The spline algorithm is asymptotically optimal, i.e.*,

$$\mathrm{SPLINE}(\epsilon) = \Theta(\mathrm{COMP}(\epsilon)) = \Theta(\epsilon^{-N/(m+r)}) \qquad \text{as } \epsilon \to 0.$$

(2) *If $k \ge 2m - 1 + r$, the FEM is asymptotically optimal, i.e.*,

$$\mathrm{FEM}(\epsilon) = \Theta(\mathrm{COMP}(\epsilon)) = \Theta(\epsilon^{-N/(m+r)}) \qquad \text{as } \epsilon \to 0.$$

(3) *If $k < 2m - 1 + r$, then*

$$\frac{\mathrm{FEM}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \frac{\mathrm{FEM}(\epsilon)}{\mathrm{SPLINE}(\epsilon)} = \Theta\left(\left(\frac{1}{\epsilon}\right)^{\lambda N}\right) \qquad \text{as } \epsilon \to 0,$$

where

$$\lambda = \frac{1}{k+1-m} - \frac{1}{m+r} > 0, \tag{6.5}$$

so that

$$\lim_{\epsilon\to 0} \frac{\mathrm{FEM}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \lim_{\epsilon\to 0} \frac{\mathrm{FEM}(\epsilon)}{\mathrm{SPLINE}(\epsilon)} = \infty. \ \square$$

Hence when k is too small for a given value of r, there is an infinite asymptotic penalty for using the FEM instead of the spline algorithm. Theorem 6.1 implies that there is an $\epsilon_0 > 0$ such that

$$\mathrm{SPLINE}(\epsilon) < \mathrm{FEM}(\epsilon) \qquad \text{for } 0 < \epsilon < \epsilon_0. \tag{6.6}$$

What is the value of ϵ_0? If ϵ_0 is unreasonably small, it may turn out that it is more reasonable to use the FEM for "practical" values of ϵ. We determine the value of ϵ_0 for a model problem in

EXAMPLE 6.1. Let $N = 1$, $\Omega = (0, \pi)$, $m = 1$, $r = 1$, $H_E^1(\Omega) = H_0^1(0, \pi)$, and consider the bilinear form B: $H_0^1(0,\pi) \times H_0^1(0,\pi) \to \mathbb{R}$ defined by

$$B(v, w) = \int_0^{\pi} v'w' \qquad \forall v, w \in H_0^1(0, \pi). \tag{6.7}$$

Hence for $f \in H^1(0, \pi)$, $u = Sf$ is the variational solution to the problem

$$\begin{aligned} -u'' &= f \qquad \text{in } (0, \pi) \\ u(0) &= u(\pi) = 0. \end{aligned} \tag{6.8}$$

We choose $\mathscr{S}_n$ to be the n-dimensional subspace of $H_0^1(0, \pi)$ consisting of piecewise linear polynomials with nodes at $x_j = j\pi/(n+1)$ for $0 \le j \le n + 1$, so that $k = 1$. Moreover, since any function in $\mathscr{S}_n$ must vanish at the endpoints of $[0, \pi]$, we see that $\dim \mathscr{S}_n = n$.

We first give a lower bound on the error $e(\varphi_n, N_n)$ of the nth FEM. Define f: $[0, \pi] \to \mathbb{R}$ by

$$f(x) = \frac{1}{\sqrt{\pi}} \qquad \forall x \in [0, \pi]. \tag{6.9}$$

Then

$$\|f\|_1 = 1$$

and $Sf = u$, where

$$u(x) = \frac{1}{2} x \left(\sqrt{\pi} - \frac{x}{\sqrt{\pi}} \right). \tag{6.10}$$

Let $\widetilde{u_n}$ denote the $\mathscr{S}_n$-interpolate of u. Using standard techniques (see, e.g., [2, 8, 9]) it is easy to show that

$$e(\varphi_n, N_n) \geq \inf_{s \in \mathscr{S}_n} |Sf - s|_1 = |u - \widetilde{u_n}| = \frac{\pi}{\sqrt{12}\,(n+1)}, \tag{6.11}$$

giving the desired lower bound on the error of the FEM.

Now we can find a lower bound on FEM(ϵ). Let n be chosen so that $e(\varphi_n, N_n) \leq \epsilon$. Then (6.11) yields

$$n \geq \frac{\pi}{\sqrt{12}}\epsilon^{-1} - 1. \tag{6.12}$$

Using (6.12) and the inequality

$$\operatorname{cost}(\varphi_n, N_n) \geq (c+1)n - 1, \tag{6.13}$$

(see [10, Chap. 5] for further discussion), we have

$$\operatorname{FEM}(\epsilon) \geq (c+1)\left(\frac{\pi}{\sqrt{12}}\epsilon^{-1} - 1\right) - 1. \tag{6.14}$$

Next, we need to find an upper bound on the error $e(\varphi_n^s, N_n)$ of the spline algorithm φ_n^s using finite element information N_n. Since the spline algorithm φ_n^s is an optimal error algorithm using N_n, it suffices to compute the radius of FEI. Let $z \in \ker N_n \cap F$. Letting P_n denote the orthogonal projector of $L_2(0, \pi)$ onto $\mathscr{S}_n$ and $\tilde{}$ denote the $\mathscr{S}_n$-interpolation operator, we find

$$\begin{aligned}
\|Sz\|_B^2 &= B(Sz, Sz) = (z, Sz)_0 = (z, Sz - P_n Sz)_0 \\
&= \left(z - \widetilde{z_n}, Sz - P_n Sz\right)_0 \\
&\leq \|z - \widetilde{z_n}\|_0 \|Sz - P_n Sz\|_0 \leq \|z - \widetilde{z_n}\|_0 \|Sz - (\widetilde{Sz})_n\|_0.
\end{aligned} \tag{6.15}$$

Now Theorem 2.4 of [8] states that for any $v \in H_0^1(0, \pi)$, we have

$$\|v - \widetilde{v_n}\|_0 \leq \frac{1}{n+1}|v|_1. \tag{6.16}$$

Since $z \in F$ (so that $|z|_1 \leq 1$) and $\|\cdot\|_B = |\cdot|_1$, we may use (6.15) and (6.16) to find

$$\|Sz\|_B \leq \left(\frac{1}{n+1}\right)^2 \qquad \forall z \in \ker N_n \cap F, \tag{6.17}$$

which yields

$$e(\varphi_n^s, N_n) = r(N_n) = \sup_{z \in \ker N_n \cap F} \|Sz\|_B \le \left(\frac{1}{n+1}\right)^2. \tag{6.18}$$

Now we can find an upper bound on SPLINE(ϵ). Let

$$n = \epsilon^{-1/2} - 1. \tag{6.19}$$

Then (6.19) yields that

$$e(\varphi_n^s, N_n) \le \epsilon. \tag{6.20}$$

From (6.20) and the inequality

$$\mathrm{cost}(\varphi_n^s, N_n) \le (c+2)n - 1 \tag{6.21}$$

(see [10, Chap. 5] for further discussion), we have

$$\mathrm{SPLINE}(\epsilon) \le (c+2)(\epsilon^{-1/2} - 1) - 1. \tag{6.22}$$

We now wish to find $\epsilon_0 = \epsilon_0(c)$ such that (6.6) holds. Using (6.14) and (6.22), we find that ϵ_0 may be chosen as the smallest positive solution of

$$(c+1)\left(\frac{\pi}{\sqrt{12}}\epsilon_0^{-1} - 1\right) = (c+2)\left(\epsilon_0^{-1/2} - 1\right). \tag{6.23}$$

Some algebra yields

$$\epsilon_0 = \epsilon_0(c) = \left(\frac{1}{2}c + 1 - \sqrt{\left(\frac{1}{2}c+1\right)^2 - \frac{\pi}{\sqrt{12}}(c+1)}\right)^2. \tag{6.24}$$

We now examine the value of $\epsilon_0(c)$ under various assumptions on the cost c of evaluating a linear functional, noting that $\epsilon_0(c)$ increases with (nonnegative) c. Clearly, $c \ge 0$, so that

$$\epsilon_0(c) \ge \epsilon_0(0) = \left(1 - \sqrt{1 - \frac{\pi}{\sqrt{12}}}\right)^2 \doteq 0.4829. \tag{6.25}$$

This tells us that no matter what we assume about the cost c of evaluating a linear functional, (6.6) holds for all ϵ less than (roughly) 0.4829. Next, we assume that $c \ge 1$, i.e., that evaluating a linear functional is at least as hard as an arithmetic operation (it would be hard to imagine otherwise). Under this assumption, we find that

$$\epsilon_0(c) \geq \epsilon_0(1) = \left(\frac{3}{2} - \sqrt{\frac{9}{4} - \frac{\pi}{\sqrt{3}}} \right)^2 \doteq 0.7048. \tag{6.26}$$

Finally, it is reasonable to assume that c is very large, i.e., that evaluating a linear functional is *much* harder than an arithmetic operation [10, p. 85]. One may check that

$$\lim_{c \to \infty} \epsilon_0(c) = \frac{\pi^2}{12} \doteq 0.8225, \tag{6.27}$$

giving an estimate of $\epsilon_0(c)$ for large values of c. □

Based on this example, it seems reasonable to conjecture that for any regularly elliptic boundary-value problem, (6.6) will hold, where ϵ_0 is sufficiently large to be of interest. We suspect that such a result will be difficult to establish, since "sufficiently large" is a subjective criterion. However, there is an even greater source of difficulty. Determining ϵ_0 for a given problem requires us to know explicit values of the order-of-magnitude constants in Theorems 4.1 and 4.2. Such constants are usually difficult to determine, except for special model problems such as the one studied in Example 6.1.

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